Stability, Instability, and Terminal Attitude Motion of a Spinning, Dissipative Spacecraft

Thomas R. Kane* and David A. Levinson†
Stanford University, Stanford, Calif.

This paper deals with the following questions: What is the physical significance of the words "stability" and "instability" as used in connection with attitude motions of a spacecraft carrying a passive nutation damper? What is the nature of the terminal motion of the spacecraft? To answer these questions, stability criteria are formulated in analytical terms, numerical integrations of differential equations governing system behavior are performed, and both qualitative and quantitative descriptions of terminal motions are formulated. Numerical results are displayed in the form of time-plots of the angle between a spacecraft-fixed line and a space-fixed line, and these reveal that instability may be attributable to a variety of causes and may manifest itself in a variety of ways. Two types of terminal motion associated with instabilities arising from energy dissipation are discussed in detail.

Introduction

IN technical literature concerned with spacecraft attitude control, the word "stability" frequently serves as an abbreviation for the phrase "stability and instability". This semantic fact may be a reflection of the practical reality that stability is generally regarded as desirable, whereas instability is regarded as objectionable. Nevertheless, whatever insights one can acquire regarding *unstable* motions are not without value, for they enhance one's understanding of the entire subject and thus contribute to one's ability to ensure stable performance. Moreover, even an unstable motion may, at times, be of practical interest. The present paper is intended to provide new insights by discussing the behavior of a particular system of practical significance, namely a spinning spacecraft carrying a rudimentary passive nutation damper. For this system, quite simple stability criteria[1] can be formulated explicitly. It is our purpose to explore the physical significance of these criteria by examining both stable and unstable motions of the spacecraft.

Stability Criteria

In Fig. 1, B designates a rigid body carrying a particle P that is attached to a spring S and a dashpot D. Y_1, Y_2, Y_3 are principal axes of inertia of B for the mass center B^* of B (hereafter called central principal axes); P is constrained to move on a line parallel to Y_1; and S is presumed to be undeformed when P lies on Y_2. In the absence of external forces, this system can perform a motion of "simple spin"; that is, it can move in such a way that P remains on Y_2 and the orientation of Y_1 in an astronomical reference frame A remains fixed while the angular velocity of B in A has a constant magnitude Ω and is permanently parallel to Y_1. If such a motion is disturbed at some instant of time t, say $t=0$, then the orientation of Y_1 in A and the distance q between P and Y_2 (see Fig. 1) generally vary with time for $t>0$, and the simple spin under consideration is said to be unstable if one cannot keep both q and the departure of Y_1 from its original orientation arbitrarily small for $t>0$ by making the disturbance sufficiently small. (The purpose of the nutation damper formed by P, S, and D is, of course, to attenuate changes in the orientation of Y_1.) Now, three conditions, violation of any one of which guarantees instability, can be formulated in terms of the spin speed Ω and the various system parameters by using a method[1] described previously in this journal. But, as is so frequently the case when one is working with stability criteria, awareness of the presence of an instability does not, per se, shed any light on the physical, and hence the practical significance of the instability. Specifically, since instability does not necessarily indicate unlimited departure of Y_1 from its original orientation, one does not learn from a stability analysis alone how a disturbance of a simple spin at $t=0$ affects the motion of Y_1 in A for $t>0$, and one is thus left wondering whether or not instability should, in fact, be a matter of concern. One can, however, explore this by integrating suitable differential equations numerically, and we shall describe the results of such integrations following a discussion of the aforementioned instability criteria.

In addition to the spacecraft spin speed Ω, the instability criteria involve parameters characterizing the body B and the nutation damper components P and S. For our purposes, it is convenient to regard B as a uniform, rectangular block having a mass density ρ and sides of lengths L_1, L_2, L_3 (see Fig. 1); to let P have a mass ν times that of B; and to choose for S a linear spring with a spring constant σ. If b is the distance between Y_1 and the axis of S (see Fig. 1), and three quantities u_1, u_2, u_3 are defined as

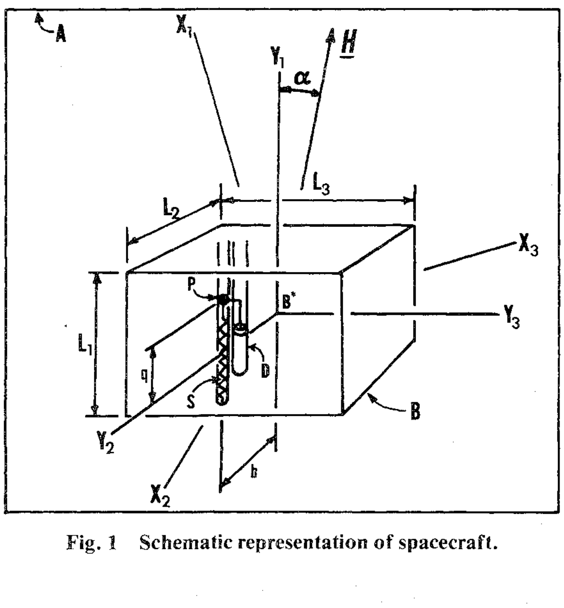

Fig. 1 Schematic representation of spacecraft.

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*Professor of Applied Mechanics.

†Research Assistant, Department of Applied Mechanics. Student Member AIAA.

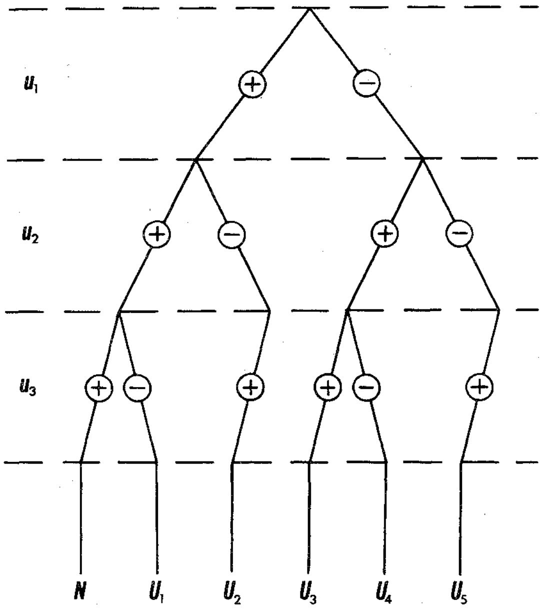

Fig. 2 Instability conditions.

$$u_1 = L_3 - L_1,\ u_2 = (L^2_2 - L^2_1)\ (1 + \nu) + 12\nu b^2 \qquad (1)$$

$$u_3 = \sigma(1+\nu)/\nu\rho L_1 L_2 L_3 \Omega^2 - 12\nu b^2/u_2 \qquad (2)$$

then the procedure described in Ref. 1 leads directly to the conclusion that the simple spin under consideration is unstable whenever at least one of u_1, u_2, u_3 is negative. (Precisely the same conditions, obtained by a different method, were recently stated by Sarychev[2].) The two conditions $u_1 < 0$, $u_2 < 0$ may be termed "non-maximum inertia" criteria, for satisfaction of either one guarantees that the angular velocity vector characterizing the simple spin is parallel to a line other than the axis of maximum central moment of inertia of the system formed by B and P. The condition $u_3 < 0$ can be regarded as a "compliance" criterion, because it can be satisfied only when the spring stiffness σ is sufficiently small.

All three instability conditions can be violated simultaneously in only one way, indicated in Fig. 2 by the letter N (standing for "not unstable"); but one or more of the conditions can be satisfied in the five ways associated with the letters U_1, ..., U_5 in Fig. 2. (Since u_3 involves u_2 in such a way that $u_2 < 0$ implies $u_3 \geq 0$, the two combinations $u_1 > 0$, $u_2 < 0$, $u_3 < 0$ and $u_1 < 0$, $u_2 < 0$, $u_3 < 0$ cannot occur, and are therefore omitted from Fig. 2.) Let us briefly examine U_1, ..., U_5 to explain why some of these merit closer scrutiny than do others.

In the case of U_1, one is dealing with a spacecraft for which the simple spin under consideration would not be unstable if P were rigidly attached to B; for, with $u_1 > 0$ and $u_2 > 0$, the spin axis is parallel to the axis of maximum central moment of inertia (when P lies on Y_2), and for a rigid body this guarantees marginal stability. Hence U_1 represents de-stabilization attributable to the nutation damper, a matter of potential practical concern, and we shall examine the nature of this destabilization in detail‡. U_2, U_3, and U_4 all correspond to motions that would be unstable even if P were fixed on B, because the spin axis is parallel to a central principal axis that is neither the axis of maximum, nor the axis of minimum central moment of inertia. For a rigid body, the associated instability is well known to manifest itself in the form of large excursions of Y_1, so we shall not pursue U_2, U_3, and U_4 further. By contrast, U_5 again involves a spacecraft in connection with which the mobility of P is of crucial importance, the spin axis here being parallel to the axis of minimum central moment of inertia, so that, if P were fixed on B, one would have marginal stability. Moreover, instability now cannot be blamed on excessive compliance (since $u_3 > 0$), but may be viewed as representing destabilization arising from energy-dissipation. This topic has long been of interest,[5-10] so we shall return also to U_5.

‡The possibility of this occurrence was pointed out by Pringle[3] in 1966. A Note dealing with the same subject[4] appeared in this journal recently.

Stable and Unstable Motions

After assigning numerical values to all system parameters, one may proceed as follows: write dynamical differential equations governing q and the Y_1, Y_2, Y_3 measure numbers, ω_1, ω_2, ω_3, of the angular velocity ω of B in A; write kinematical differential equations involving ω_1, ω_2, ω_3 and the elements of the direction cosine matrix $[C_{ij}]$ that relates Y_1, Y_2, Y_3 to axes X_1, X_2, X_3 fixed in A and initially coincident with Y_1, Y_2, Y_3, respectively; integrate all differential equations simultaneously, using the initial conditions $q = \dot{q} = 0$, $\omega_1 = \Omega$, $\omega_2 = \epsilon\Omega$, $\omega_3 = 0$, and $[C_{ij}] = I$, the 3×3 identity matrix ($\omega_2 = \epsilon\Omega$ represents a disturbance of a simple spin); calculate for discrete instants of time during the integration the angle α (see Fig. 1) between Y_1 and the central angular momentum vector H of the system (which remains fixed in A for $t > 0$); and plot α vs t.

To set the stage for subsequent study of U_1 and U_5, we first consider N in Fig. 2, taking $\rho = 2760$ kg m^{-3} (the mass density of aluminum), $L_1 = 1.200$ m, $L_2 = 1.225$ m, $L_3 = 1.300$ m, which concludes the description of B. Next, we set $\nu = 0.01$, so that the mass of P is 1% of that of B, and then complete the specification of the nutation damper by letting $\sigma = 52.744$ N m^{-1} and $\delta = 105.487$ N sec m^{-1}, where δ is a constant that characterizes the dashpot D in the following sense: the force exerted by D on P is presumed to have a magnitude of $\delta|\dot{q}|$. The numerical values of σ and δ here used are such that the oscillator formed by P, S, and D has a natural (undamped) circular frequency of 1 rad sec^{-1} and is critically damped. Finally, we set $b = 1$ m (see Fig. 1) and $\Omega = 1$ rad sec^{-1}; and, noting that, by Eqs. (1) and (2), we now have $u_1 = 0.100$ m, $u_2 = 0.181$ m^2, $u_3 = 0.348$, so that we are indeed dealing with a motion that violates all three instability criteria (see Fig. 2), we perform the numerical integration described previously, taking $\epsilon = 0.1$. This leads to the curve labeled $N(.1)$ in Fig. 3a, which shows that α (see Fig. 1) decays with increasing t. The nutation damper is thus performing its intended function. (It is interesting to note that this satisfactory behavior occurs when the oscillator has a natural frequency equal to the spin speed of the spacecraft.) Next, we increase the compliance of S by decreasing σ to 26.372 N m^{-1}, one half of its former value, but leave all other parameter values unchanged, which has the effect of making $u_3 = -0.157$ while u_1 and u_2 retain their former (positive) values. Hence we are dealing with U_1 in Fig. 2. The same disturbance used previously ($\epsilon = 0.1$) now gives rise to the curve labeled U_1 (.1) in Fig. 3a, from which it is immediately apparent that the nutation "damper" is, in fact, acting as a nutation generator. Now, one might conjecture that the large response here encountered is attributable to an excessively large initial disturbance. To see that this is not the case, one can change ϵ from 0.1 to 0.05 and then reexamine both N and U_1, which was done to produce the curves N (.05) and U_1 (.05) in Fig. 3a. The four curves in Fig. 3a illustrate a fundamental proposition: When a motion is stable, reducing a disturbance has the effect of reducing departures from the nominal motion; but when a motion is unstable, reducing a disturbance merely defers such departures. Thus, N (.05) lies below N (.1) for all t, whereas U_1 (.05) attains values comparable to the largest values of U_1 (.1), albeit more slowly.

An instability need not manifest itself so unequivocally as did the one just examined. For example, if the dimensions of B are changed to $L_1 = 0.500$ m, $L_2 = 1.200$ m, $L_3 = 3.185$ m,

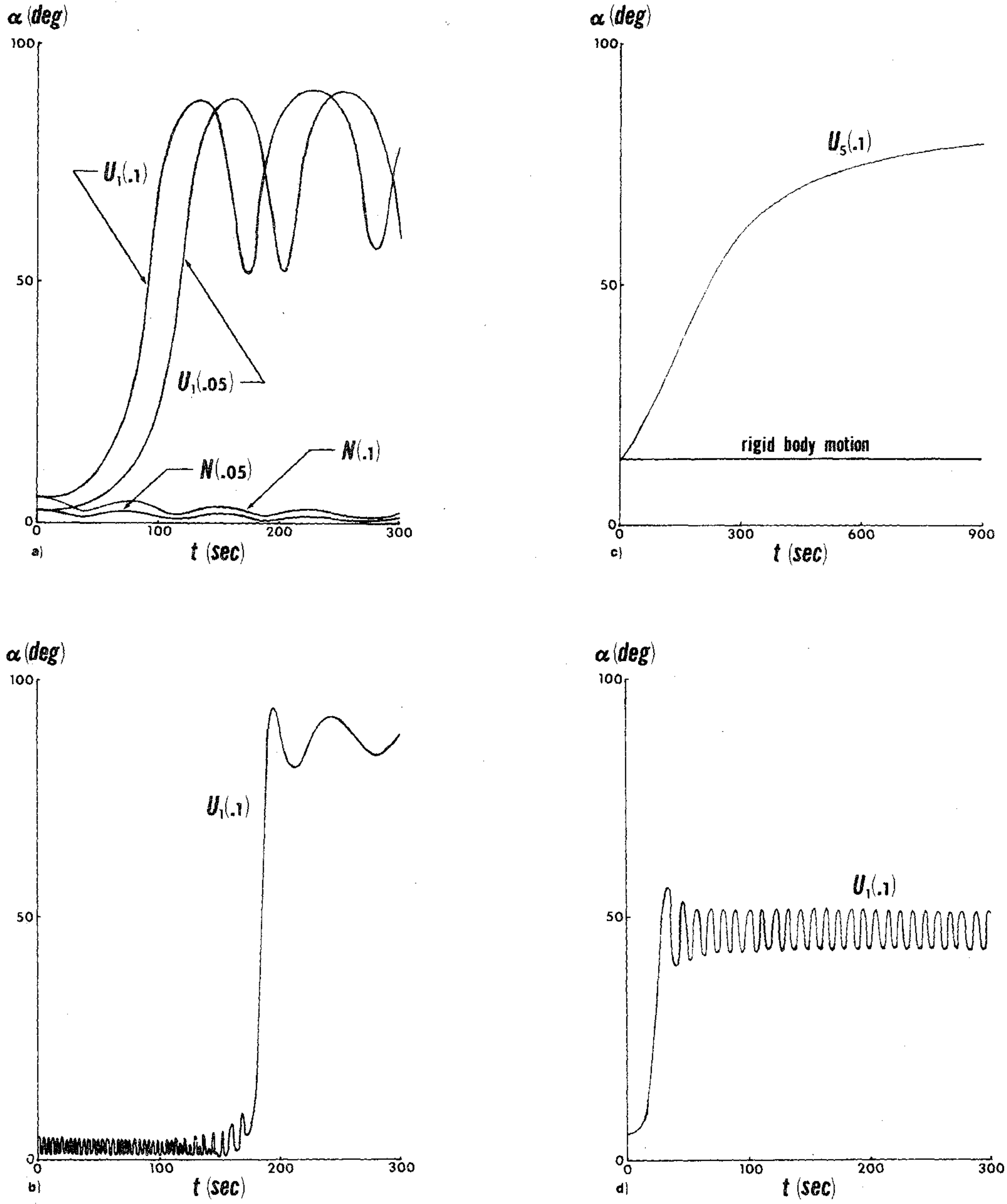

Fig. 3 Time dependence of α: a) Stable and unstable max axis spins; b) Potentially deceptive max axis spin; c) Unstable min axis spin; d) Unstable max axis spin.

and the spring constant is changed to $\sigma=0.527$ N m^{-1}, but ρ, ν, δ, b, and Ω are left unaltered (B has the same mass as heretofore, and the spring is one hundred times more compliant than in the preceding case), then $u_1=2.685$ m, $u_2=1.322$ m^2, $u_3=-0.081$, so that one is still dealing with U_1 in Fig. 2. But now, for the same disturbance as before ($\epsilon=0.1$), one obtains the rather more complex response depicted in Fig. 3b, one interesting feature of which is that α remains quite small for the first two minutes of the motion, giving the appearance of, at least, marginal stability, but then grows very rapidly. This plot illustrates a second fundamental fact: When a motion is of practical interest for only a limited time interval, instability of the motion may be tolerable, because objectionably large departures from the nominal motion may occur only subsequent to the time of interest.

The instability associated with U_5 in Fig. 2 arises, for example, if $L_1=2.000$ m, $L_2=0.948$ m, $L_3=1.008$ m while ρ, ν, σ, δ, b, and Ω have the values used earlier in connection with N. For these values of the system parameters, B has the same mass as heretofore; the central inertia ellipsoid of the system formed by B and P is a prolate spheroid when P is on Y_2 (the central principal moments of inertia have the values 894 kg m^2 and 2205 kg m^2), and the axis of revolution of this spheroid is parallel to Y_1, so that the nominal motion under consideration is one during which the angular velocity vector of B is parallel to the axis of minimum moment of inertia; and $u_1=-0.992$ m, $u_2=-3.012$ m^2, $u_3=1.050$. With $\epsilon=0.1$, one here obtains the curve labeled $U_5(.1)$ in Fig. 3c, which shows that energy dissipation now causes B to go into a "flat spin"; that is, the Y_1 axis approaches prependicularity with the angular momentum vector. The claim that energy dissipation plays a major role is particularly strong here because, if P were fastened to Y_2, which would eliminate energy dissipation, the rigid body formed by B and P would perform a well known motion of precession accompanied by spin§, and this would proceed in such a way that α remains at all times equal to its initial value, as indicated by the horizontal line in Fig. 3c.

§Such motions are described later in more details in a different context.

Before leaving this case, it is worth mentioning that integration results lend themselves particularly well to a delineation of the terminal motion. They reveal that not only is α approaching the value of ninety degrees, but the system tends toward a rigid body motion of simple spin with a terminal angular speed of 0.418 rad sec^{-1}. This may be deduced as follows: $|q_{max}|$, which never exceeds 0.305 m during the first fifteen minutes of the motion, tends towards zero (e.g., for $t \geq 800$ sec, $|q_{max}| < 0.165$ m), showing that the system is becoming rigid. The sum of the system's kinetic and potential energies goes monotonically from 457.973 N m at $t=0$ to 203.007 N m at $t=900$ sec. Now, if the system were performing a rigid body motion of simple spin about an axis perpendicular to Y_1, its total energy would be given by $E=H^2/2I$, where H is the magnitude of $\boldsymbol{H}$ and I is the moment of inertia about a central axis normal to Y_1. For the case at hand, $E=192.215$ N m. Thus the postulated motion is being approached. Finally, during this motion, $|\omega|$ would have the value $|\omega| = H/I = 0.418$ rad sec^{-1}, and integration results show that $|\omega|$ decreases monotonically, reaching the value $|\omega| = 0.455$ rad sec^{-1} at $t=900$ sec.

The curves labeled $U_1(.1)$ in Figs. 3a and 3b differ so markedly from the curve $U_5(.1)$ in Fig. 3c that one is led to wonder about the terminal motion associated with U_1. To investigate this, we take $\sigma=0.527$ Nm^{-1}, but assign to all other parameters the values used in connection with $U_1(.1)$ in Fig. 3a, which leads to $u_1=0.100$ m, $u_2=0.181$ m^2, $u_3=-0.652$. The reason for the present choice of a value for σ is that this speeds up attainment of a relatively steady state of affairs, as represented by the curve $U_1(.1)$ in Fig. 3d for, say, $t>250$ sec. What we seek is a detailed description of the behavior of the system.

Consider once again the motion of a torque-free axisymmetric rigid body. This proceeds as follows: if Z is the symmetry axis of the body, then the plane formed by Z and by a line X that intersects Z and is parallel to $\boldsymbol{H}$ rotates about X with a constant angular speed p, called the precession speed; the angle θ between X and Z remains constant; and the body rotates about Z relative to the X-Z plane with an angular speed s, called the spin speed. The quantities p, s, and θ are given by

$$p=|\boldsymbol{H}|/I, \quad s=(I-J)\omega/I, \quad \theta=\cos^{-1}[Js/(I-J)p] \qquad (3)$$

where I and J are respectively the central transverse moment of inertia and the axial moment of inertia of the body and ω is the projection on Z of the inertial angular velocity of the body. Now let Y be a line that is fixed in the body, passes through the mass center, and forms an angle β with Z; and let γ be the angle between X and Y. Then $|\gamma|$ fluctuates (with a circular frequency s) between $\gamma=\theta+\beta$, and $\gamma=\theta-\beta$, and a plot of γ versus t must have the same general appearance as does the curve $U_1(.1)$ in Fig. 3d for $t>250$ sec. This suggests that the system to which Fig. 3d applies is performing a motion resembling that of some torque-free, axi-symmetric body. This is, indeed, the case, as will now be shown.

Integration results reveal that $|q|$ grows slowly and monotonically for $t>250$ sec. For example, at $t=250$ sec, $|q|=13.271$ m, and at $t=300$ sec., $|q|=13.323$ m. Hence, the central principal moments of inertia of the system also vary slowly, and we may confine our attention to a particular value of t, say $t=300$ sec. Now, at this instant, the central principal moments of inertia are found to have the values 1402 kg m^2, 10614 kg m^2, and 10698 kg m^2, and it is immediately evident that the central inertia ellipsoid of the system is very nearly a prolate spheroid. (The explanation for this is readily at hand: $|q|$ is so large that P, despite its relatively small mass, contributes substantially to each of the two transverse moments of inertia, which accounts both for their near equality and for the prolateness of the spheroid.) We now identify the symmetry axis of this spheroid with Z, assign to I and J the values $I=(10614+10698)/2$ kg m^2 and $J=1402$ kg m^2. Next, we let Y_1 play the role of Y, and calculate the angle between the symmetry axis of the spheroid and Y_1, which must then correspond to β. This turns out to have the value $\beta=4.305$ deg. Furthermore, the angle α plotted as the curve $U_1(.1)$ in Fig. 3d should now correspond to γ, and should, therefore, fluctuate between $\theta+4.305$ and $\theta-4.305$ degrees, where θ is given by the third of Eqs. (3). Hence we first evaluate p and s in accordance with the first two of Eqs. (3), obtaining $p=0.137$ rad sec^{-1} and $s=0.613$ rad sec^{-1} (using $\omega=0.706$ rad sec^{-1}, as found from the numerical integration). For θ we then find $\theta=47.321$ deg, and α should, therefore, fluctuate between 51.626 deg and 43.016 deg. The numerical integration results show that, for $250 \leq t \leq 300$ sec., $\alpha_{max}=51.661$ deg and $\alpha_{min}=43.019$ deg. Moreover, the fluctuations in α have a period of approximately 10.3 sec, which agrees very well with $2\pi/s$. One can conclude, therefore, that U_1 involves spin and precession of a certain apparently axisymmetric rigid body at any given (sufficiently great) time, but two different such bodies come into play at two widely separated instants of time. However, more remains to be said, for this motion is accompanied by energy dissipation which must cease ultimately, the total available energy being finite.¶ Hence, the motion must tend toward a quasi-rigid body motion, that is, a motion during which P remains at rest relative to B. Moreover, this quasi-rigid body motion must be a simple spin, that is, a motion during which the angular velocity vector has a time-independent orientation relative to the body (otherwise, P is subjected to the action of variable inertia forces); and this can occur if and only if the angular velocity vector remains parallel to a central principal axis of inertia of the entire system. Finally, the principal axis in question must be the axis of maximum central moment of inertia, in order that the "external" stability criteria of Ref. 1 be satisfied; and the total energy associated with the motion must be smaller than the total initial energy. Now, for the parameter values and initial conditions used to construct Fig. 3d, there exists precisely one motion that satisfies all of these requirements. The associated values of q and α are -15.845 m and 86.381 deg, respectively, and the total energy is equal to 139.626 Nm. The numerical integration performed to generate Fig. 3d shows that, at $t=300$ sec., q and the total energy are equal to -13.323 m and 450.274 Nm, respectively. Evidently, the terminal motion is being approached very slowly.

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